Rolling Ball Simulation

Define the plant and controller, calculate the closed loop gains.

$$G_a(s) = \frac{K}{s^2 + d\cdot s}$$

Type 2 system with a drag or friction coefficient proportional to the velocity

$$G_c(s) = Kp + Kd\cdot s$$

A simple PD controller.

$$CLTF(s) = \frac{(Kp + Kd\cdot s)\cdot\frac{K}{s^2 + d\cdot s}}{1 + (Kp + Kd\cdot s)\cdot\frac{K}{s^2 + d\cdot s}}$$

Closed loop transfer function (CTLF) for combined actuator and controller transfer functions.

$$CLTF(s) = \frac{G_c(s)\cdot G_a(s)}{1 + G_c(s)\cdot G_a(s)}$$

Closed loop transfer function with the characteristic equation in the denominator.

$$CLTF(s) = \frac{(Kp + Kd\cdot s)\cdot K}{s^2 + (d + K\cdot Kd)\cdot s + K\cdot Kp}$$

Simplified closed loop transfer function

$$(s + \lambda)^2 \text{ expand } \rightarrow s^2 + 2\cdot s\cdot\lambda + \lambda^2$$

Desired characteristic equation. In this case a critically damped response is desired.

Equate the powers of s for the coefficients of the characteristic equation of the closed loop transfer function and the desired characteristic equation. The solve for the two unknown gains using the two equations. This is simple enough to do by inspection.

Given

$$\lambda^2 = K\cdot Kp$$

Match the coefficients for the 0 power of s

$$2\cdot\lambda = K\cdot Kd + d$$

Match the coefficients for s

$$\text{Find}(Kp, Kd) \rightarrow \begin{pmatrix} \frac{\lambda^2}{K} \\ \frac{2\cdot\lambda - d}{K} \end{pmatrix}$$

These are the symbolic formulas for the proportional and derivative gains. That place both poles at $-\lambda$
This is so simple!

$$\frac{r}{s}\cdot\frac{(Kp + Kd\cdot s)\cdot K}{(s + \lambda)^2} \left| \begin{array}{l} \text{substitute}, Kp = \frac{\lambda^2}{K} \\ \text{substitute}, Kd = \frac{2\cdot\lambda - d}{K} \\ \text{invlaplace}, s \\ \text{simplify} \\ \text{collect}, r, K, e^{-\lambda\cdot t} \end{array} \right. \rightarrow \left[\left[(-1) + \lambda\cdot t - t\cdot d\right]\cdot e^{(-\lambda)\cdot t} + 1\right]\cdot r$$

Find the response as a function of time by taking the inverse Laplace transform and converting to the time domain.

$$pos(t) = \left[\left[(-1) + \lambda\cdot t - t\cdot d\right]\cdot e^{(-\lambda)\cdot t} + 1\right]\cdot r$$

The inverse Laplace response is a little different than the differential equation response.

Start with energy balance

$$m\cdot g\cdot h = \frac{1}{2}\cdot m\cdot v^2 + \frac{1}{2}I\cdot\omega^2$$

Substitute for inertia and angular velocity

$$m\cdot g\cdot h = \frac{1}{2}\cdot m\cdot v^2 + \frac{1}{2}I\cdot\omega^2 \ \left|\begin{array}{l} \text{substitute}, I = \frac{2}{3}\cdot m\cdot r^2 \\ \text{substitute}, \omega = \frac{v}{r} \end{array}\right. \rightarrow m\cdot g\cdot h = \frac{5}{6}\cdot m\cdot v^2$$

Solve for v^2

$$m\cdot g\cdot h = \frac{5}{6}\cdot m\cdot v^2 \ \text{substitute}, v^2 = 2\cdot a\cdot d \ \rightarrow m\cdot g\cdot h = \frac{5}{3}\cdot m\cdot a\cdot d$$

Solve for a, the acceleration

$$m\cdot g\cdot h = \frac{5}{3}\cdot m\cdot a\cdot d \ \text{solve}, a \ \rightarrow \frac{3}{5}\cdot g\cdot\frac{h}{d}$$

h is the height lost and d is the distance rolled down hill so $h/d=\sin(\theta)$. g=9.8

$$a = \frac{3}{5}\cdot 9.8\cdot\sin(\theta) \ \text{float}, 6 \ \rightarrow a = 5.88000\cdot\sin(\theta)$$

$K := 5.88$

Approximate $\sin(\theta)$ with θ for small angles

$$a = K\cdot\theta$$

Simulate the rolling ping pong ball

$\lambda := 2\cdot 2\pi$ $\quad\lambda = 12.566$ two closed loop poles at $-\lambda$

$\Delta t := 0.01$ Loop time, seconds

$$y := \begin{pmatrix} x \leftarrow -0.1 \\ v \leftarrow -0.1 \end{pmatrix}$$

Initial position in meters and velocity in meters per second

$d := 0.1$

$Kp := \frac{\lambda^2}{K}$ $\quad Kp = 26.856$ proportional gain. radians per meter of error

$Kd := \frac{2\cdot\lambda - d}{K}$ $\quad Kd = 4.257$ derivative gain. radian/(meter/second error)

Assign some real numbers for a table tennis ball rolling on a flat surface

$M := 0.0027 \cdot kg$		mass of the table tennis ball
$R := 0.02 \cdot m$		radius of the table tennis ball
$I := \frac{2}{3} \cdot M \cdot R^2$	$I = 7.2 \times 10^{-7}\, m^2 \cdot kg$	inertia of the table tennis ball
$k := \frac{I}{M \cdot R^2}$	$k = 0.667$	unitless. basically the 2/3 of the M*R^2
$K := \frac{9.8}{1 + k}$	$K = 5.88$	Combine and simplify. The units are $\frac{m}{s^2}$
$\lambda := 1 \cdot 2\pi$	$\lambda = 6.283$	two closed loop poles at -λ
$\Delta t := 0.01$		Loop time, seconds
$y := \begin{pmatrix} x \leftarrow -0.1 \\ v \leftarrow -0.1 \end{pmatrix}$		Initial position in meters and velocity in meters per second
$Kp := \frac{\lambda^2}{K}$	$Kp = 6.714$	proportional gain
	$Kd = 4.257$	derivative gain

Feed Forwards, not used

$Kv = \frac{d}{K}$	Velocity feed forward
$Ka = \frac{1}{K}$	Acceleration feed forward

$$D(t,y) := \left| \begin{array}{l} \begin{pmatrix} x \\ v \end{pmatrix} \leftarrow y \\ \theta \leftarrow Kd \cdot v + Kp \cdot x \\ a \leftarrow -K \cdot \sin(\theta) - d \cdot v \\ \begin{pmatrix} x' \\ v' \end{pmatrix} \leftarrow \begin{pmatrix} v \\ a \end{pmatrix} \end{array} \right.$$

Actual position, actual velocity and PID output

Acceleration feed forward

Integrate actual velocity and acceleration

Rolling Ball Simulation

$N := \dfrac{2}{\Delta t}$ $\quad N = 200$ Number of iterations for five seconds

$S := \mathrm{rkfixed}(y, 0, N \cdot \Delta t, N, D)$ Solution

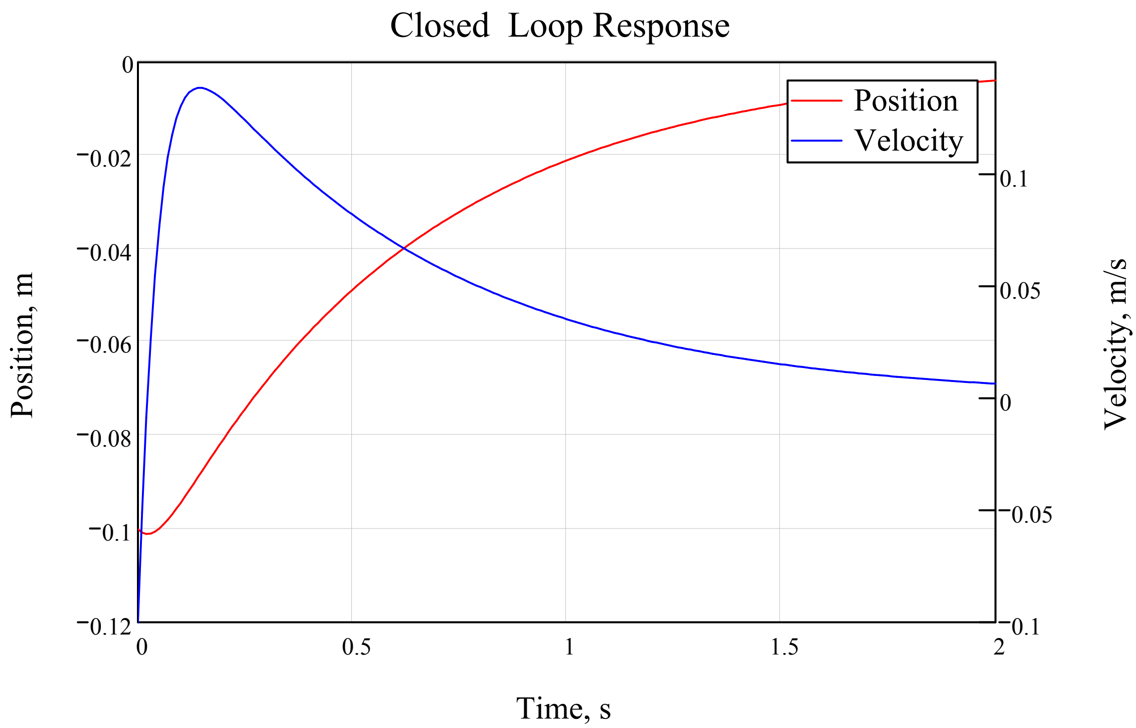
$time := S^{\langle 0 \rangle}$ seconds

$pos := S^{\langle 1 \rangle}$ meters

$vel := S^{\langle 2 \rangle}$ meters per second

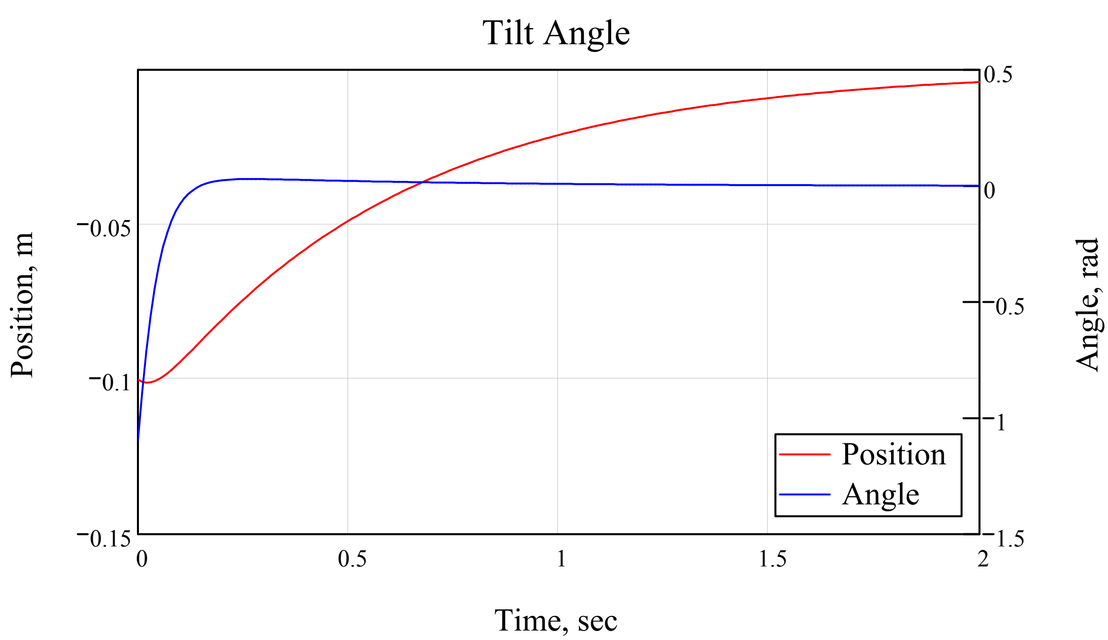
S =

	0 (time)	1 (position)	2 (velocity)
0	0	-0.1	-0.1
1	0.01	-0.101	-0.051
2	0.02	-0.101	$-8.557 \cdot 10^{-3}$
3	0.03	-0.101	0.026
4	0.04	-0.101	0.055
5	0.05	-0.1	0.077
6	0.06	-0.099	0.094
7	0.07	-0.098	0.108
8	0.08	-0.097	0.118
9	0.09	-0.096	0.125
10	0.1	-0.094	0.13
11	0.11	-0.093	0.134
12	0.12	-0.092	0.137
13	0.13	-0.09	0.138
14	0.14	-0.089	0.139
15	0.15	-0.088	0.139
16	0.16	-0.086	0.138
17	0.17	-0.085	0.137
18	0.18	-0.083	0.136
19	0.19	-0.082	0.135
20	0.2	-0.081	0.133
21	0.21	-0.079	0.132
22	0.22	-0.078	0.13
23	0.23	-0.077	0.128
24	0.24	-0.076	0.126
25	0.25	-0.074	0.124
26	0.26	-0.073	0.122
27	0.27	-0.072	0.12
28	0.28	-0.071	0.119
29	0.29	-0.069	0.117
30	0.3	-0.068	0.115
31	0.31	-0.067	0.113
32	0.32	-0.066	0.111
33	0.33	-0.065	0.109
34	0.34	-0.064	0.107
35	0.35	-0.063	0.106
36	0.36	-0.062	0.104
37	0.37	-0.061	0.102

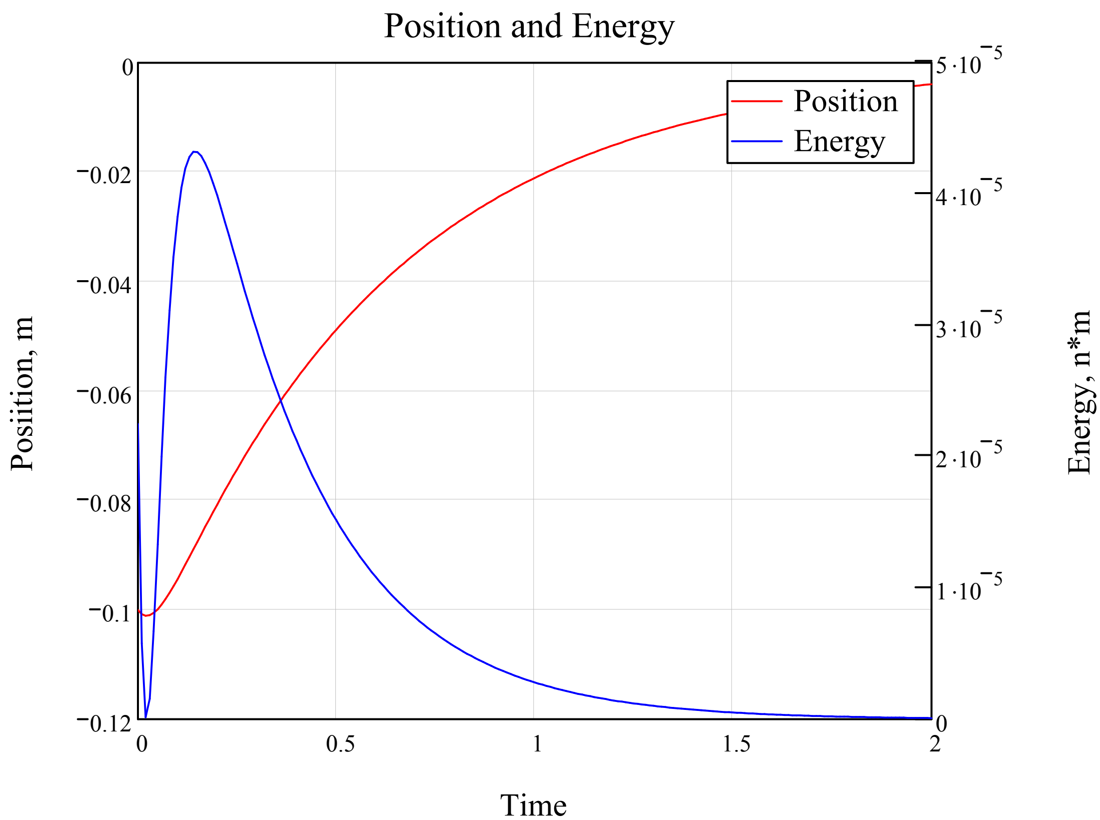


$n := 0..N$

$\theta_n := \left[Kd\cdot\left(S^{\langle 2\rangle}\right)_n + Kp\cdot\left(S^{\langle 1\rangle}\right)_n\right]$



$$E_n := \frac{1}{2}(k + 1)\cdot M\cdot\left(vel_n\right)^2$$



Rolling Ball Simulation

Simulation using Inverse Laplace Transforms

$r := 1$

$$pos(t) := \left[\left[(-1) + \lambda \cdot t - t \cdot d\right] \cdot e^{(-\lambda) \cdot t} + 1\right] \cdot r$$

$$vel(t) := \left[-e^{(-\lambda) \cdot t}\right] \cdot \left[(-2) \cdot \lambda + d + \lambda^2 \cdot t - \lambda \cdot t \cdot d\right] \cdot r$$

$t := 0, 0.01 .. 2$

It really doesn't matter what r is in this example. Without limits the response will approach r. However, the inverse Laplace transform has more overshoot.